of local waviness cause strength reductions that are only moderate. The strength reductions for the plate-stiffened columns of Ref. 2 are intermediate relative to the presently analyzed structures.

It is shown that the efficiency of lattice columns is not significantly compromised by using conventional design procedure (equating local and Euler instability loads) rather than optimum design procedures which account for initial waviness. The same trend holds true for sandwich panels, except when the initial waviness is of large amplitude. It is similarly obvious from the results of Ref. 2 that the efficiency of plate-stiffened columns is not significantly compromised by using conventional design practice.

It is therefore concluded that the effects of initial waviness on the buckling strength of built-up structures depend on the particular type of structure. Each type must be investigated separately, and many other types are still to be investigated. However, from the common trends observed here, no significant increases in efficiency are to be gained through optimizing structural proportions for a specific imperfection amplitude, regardless of the type of structure involved.

It is also observed from these studies that different types of structural elements have differing reductions in their stiffnesses for equal amplitudes of initial waviness. This accounts for the varying strength reductions among the structures analyzed here and in Ref. 2.

There still remains a wide variety of structural elements whose reductions in stiffness should be determined for various conditions of initial waviness and under various loadings. Such reductions can be important not only to the static buckling strengths but also to the vibrational and aeroelastic characteristics of the structural assemblies in which those elements are used.

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On a Numerical Sufficiency Test for Monotonic Convergence of Finite Element Models

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Finite element analyses characterized by monotonic convergence include the discipline for meaningful measurements of convergence rate and consequently economical extrapolation. Few proposers of element models guarantee monotonic convergence for their elements. Thus, a need exists for an automatic test to classify available element models. This paper describes such a test—a test that can be performed using a digital computer to guarantee that a particular element model imbues monotonicity. It describes the test and its basis. It examines seven element models for a rectangular membrane to illustrate the value of the tests. Besides confirming results already known, the application yields new data. It "proves" monotonicity for two improved models, defines the range of element proportions for which another element can be guaranteed to exhibit monotonicity, and suggests that another element is deficient. In the special case of absolutely convergent membrane displacement models, proof of monotonicity is a necessary and sufficient condition to insure that upper bound estimates of strain energy are developed. Accordingly, the test furnishes a proof of bound solutions independently of requirements on displacement continuity the element basis may or may not satisfy.

Introduction

It is important that a finite element analysis model exhibit monotonic convergence. By definition this guarantees an equal or reduced error norm for successive analyses with finer meshes. A desirable consequence of monotonic convergence is the potential for drastically improving analysis efficiency by using extrapolation techniques. In addition, monotonicity is necessary if the rate of change of the norm is used as a means of deducting the magnitude of discretization error. A need exists for an experimental test basis of monotonic convergence. Such a test would facilitate evaluating convergence characteristics of existing and proposed element models.

Guarantees of monotonic convergence for most of the elements in the literature are not currently available, perhaps because of the difficulty of proving such convergence analytically. Similarly, computer code technical documentation describing element models rarely testifies to monotonic convergence. Experience indicates that in at least two very widely used codes, elements are available which disrupt monotonic convergence of the set.

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This paper describes the basis of a sufficiency test for monotonic convergence. It illustrates its range of effectiveness by application to several membrane models. The basis covers control of discretization error. It is suitable for use without recourse to the shape functions or derivation details of the element models. It applies to displacement models of Ritz and non-Ritz elements.

Application of the test is made to linear analyses—though the concepts can be extended to nonlinear. Tests were only made for analyses in which all elements are the same type, size, and shape—though the basis is suitable for testing models of different shapes and mixed element types.

An existing basis for proving monotonic convergence of finite element models has been described by Melosh.[1] The essential idea is that if a subdivided element can give the same response prediction (displacements) everywhere in the domain as the undivided element, then monotonic convergence is guaranteed. This paper broadens this concept to discrete fields so that it is applicable independent of the existence of a shape function.

The paper is partitioned into three sections. The next section defines the test—the procedure, theoretical basis, and implications of its results. The third section illustrates use of the test on a set of rectangular membrane elements. The fourth section contains study conclusions.

Test of Monotonic Convergence

This section defines an experimental procedure for proving monotonic convergence of a given finite element displacement model. It examines the basis of the test and conclusions which can be drawn for elements which pass or fail it.

Test Procedure

The test involves five steps: Choose a set of N independent vectors, where N is the order of the stiffness matrix for the undivided element, and calculate the strain energy matrix for these vectors. This requires evaluating

$$[U_u] = 1/2[A^T][K][A] \tag{1}$$

where U_u is the matrix of energy coefficients for the undivided element, K is the element stiffness matrix, and A is the set of N independent vectors. (It is often convenient to let A be the identity matrix.)

2) Subdivide the element, write the load deflection equations, and condense, eliminating the dependency on all degrees of freedom which are not associated with boundary joints. This reduces the equations

$$[K_s](\delta'_s) = (P'_s) \tag{2}$$

where K_s is the stiffness matrix of the subdivided element and δ'_s and P'_s are the displacement and loading vectors of the subdivided elements, to

$$\begin{bmatrix} K_{uu} & K_{ub} \\ K_{bu} & K_{bb} \end{bmatrix} \begin{pmatrix} \delta'_u \\ \delta'_b \end{pmatrix} = \begin{pmatrix} P'_u \\ P'_b \end{pmatrix} \tag{3}$$

where the subscript "u" designates the degrees of freedom common to the undivided element and "b" to those uncommon degrees of freedom for the boundaries of the subdivided element.

3) Extend each eigenvector to include interpolated displacement components for all as yet undefined boundary degrees of freedom of the subdivided element. Interpolation along each side must define new joint displacements that would be consistent with rigid body response when the stiffness matrix column vector forces satisfy macroscopic equilibrium. If this is insufficient to uniquely define interpolation coefficients, make interpolation consistent with constant (nonzero) strain states as well. This step produces the displacement matrix

$$\begin{bmatrix} A'_u \\ A'_b \end{bmatrix} = \begin{bmatrix} I \\ T \end{bmatrix} [A] \tag{4}$$

where A'_u and A'_b are displacement matrices for the "u" and "b" degrees of freedom, T is the interpolation matrix, and I is the identity matrix.

4) Determine the strain energy of the subdivided model for each extended eigenvector. Thus form

$$[U_s] = [A'^T_u \quad A'^T_b] \begin{bmatrix} K_{uu} & K_{ub} \\ K_{bu} & K_{bb} \end{bmatrix} \begin{bmatrix} A'_u \\ A'_b \end{bmatrix} \tag{5}$$

where U_s is the strain energy of the subdivided element.

5) Evaluate the definiteness of

$$[C] = [U_u - U_s] \tag{6}$$

where C is the matrix of energy changes. a) If C is positive definite, monotonic convergence is guaranteed. Rigid body states have not been included. b) If C is positive semidefinite, monotonic convergence is guaranteed and some rigid body states have been included in the model. c) Otherwise, the matrix may or may not be associated with monotonic convergence.

Test Basis

The test indicates monotonic convergence directly. It shows regardless of the displacement state of the element, when subdivision cannot result in increasing the strain energy.

Selection of N vectors provides for all possible linearly independent displacement states. Condensation implies no change in the equivalent loading and thus interior joint displacements can be chosen consistent with minimizing the energy. The interpolation basis preserves zero and constant strain states, if they exist. If the difference matrix is positive semidefinite no increase in energy may occur due to the subdivision.

The interpolation basis delimits the scope of the test. Since mesh refinement of the system usually admits choosing subdivision boundary displacements to minimize energy, the test is only a sufficiency test. The consistency of interpolation with rigid body (zero strain) states implies that element matrices with forces satisfying macroscopic equilibrium are the models of interest. Consistency with constant strain states is natural because their existence is a sufficient condition for absolute convergence.[2]

For membranes and three-dimensional solids, the requirement that interpolation be consistent with rigid modes is enough to uniquely define the interpolation matrix coefficients. For plates, when displacement unknowns include joint translations and rotations, consistency with constant strain is also required for uniqueness. With higher order displacement variables, additional requirements must be specified.

Application to Membrane Models

This monotonic convergence test was applied to several rectangular membrane element models. This section describes the details and results of the membrane studies. The test was implemented by first constructing eight independent vectors of the element displacements for the single element shown in Fig. 1a. These displacement vectors were chosen so that matrix A of Eq. (1) was the identity matrix.

The single element was then divided into four equal elements as shown in Fig. 1b, and the corresponding stiffness matrix assembled. The midside node displacements were selected to be

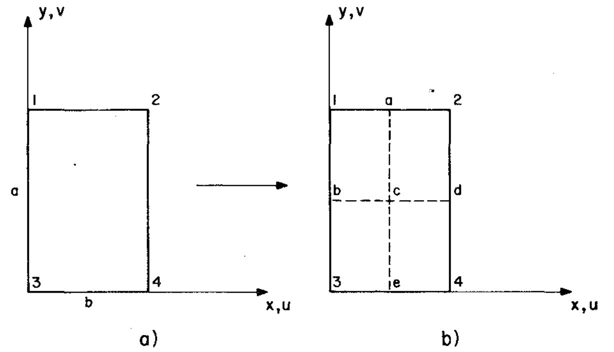

Fig. 1 Element subdivision.

Table 1 Candidate rectangular membrane element models

Symbol	Identification	Nature	Basis	Source
⊠	Hrennikoff frame	Non-Ritz	Matching constant strain behavior	Ref. 3
⧄	Two triangles	Ritz	Constant strain triangles	Ref. 4
⊠	Four triangles	Ritz	Constant strain triangles	Ref. 4
[σ]	Turner stress rectangle	Non-Ritz	Constant and linear stress	Ref. 4
[L]	Lagrange rectangle	Ritz	Linear strain (Lagrange interpolation)	Ref. 1
⊠	Unbiased triangle	Non-Ritz	Constant strain triangles	Ref. 5
⁚⁚	Finite difference	Non-Ritz	First order difference operators	

the average of the adjacent corner node displacements insuring solution when the divided element is surrounded by other elements, and providing consistent displacements for constant strain modes. To obtain the minimal strain energy with the above constraints the equations were solved for the center node displacements. Using these displacement vectors and the assembled stiffness matrix the strain energy matrix, $[U_s]$ of Eq. (5) was computed. The above procedure is equivalent to steps 1–4 of the test procedure.

Having established $[U_u]$ of Eq. (1) and $[U_s]$ of Eq. (5), the matrix of energy changes $[C]$ was computed using Eq. (6). The definiteness of this matrix was evaluated by examining its eigenvalues.

Table 1 cites the principal features of the seven element models studied. The symbols given are used on graphs later in this paper. Models are listed in the order in which they appear in the literature.

Each of the models contains constant strain states. All except the Turner stress rectangle are guaranteed to produce the exact answer to the membrane equations as the mesh interval approaches zero in the limit. The three Ritz models guarantee upper bound estimates of strain energy regardless of the mesh intervals as a consequence of their implication of continuous displacement shapes over each element and across element boundaries.

The finite difference element was developed by inserting finite difference estimates of the first derivatives using nodal displacements and side lengths, into the strain energy expression. The volume integral was carried out assuming the derivatives were constant in the quadrant adjacent to the nodal point at which they were estimated.

The test was applied to each element for panel aspect ratios ranging from 1 to 4. Results of the test are displayed graphically in Fig. 2. The vertical axis represents the eigenvalue and the horizontal, the aspect ratio. Two curves are plotted for each element corresponding to the two nonzero eigenvalues. Whenever each curve for an element lies above the horizontal axis the element is monotone. As is specified on the graph, Poisson's ratio is 1/3.

The fact that the six remaining eigenvalues are zero is explained as follows. If an element contains rigid body modes three independent displacement vectors exist which yield a zero energy difference between the undivided and subdivided element. Three additional zeros are obtained corresponding to the zero energy differences associated with constant strain states. Thus, for rigid body modes and constant strain states to exist in an element it is necessary that the energy change matrix, $[C]$, possess at least six zero eigenvalues.

Examination of the curves of Fig. 1 leads to the conclusions of Table 2. Except where noted the results are applicable for panel aspect ratios ranging from 1–4, and for a Poisson's ratio

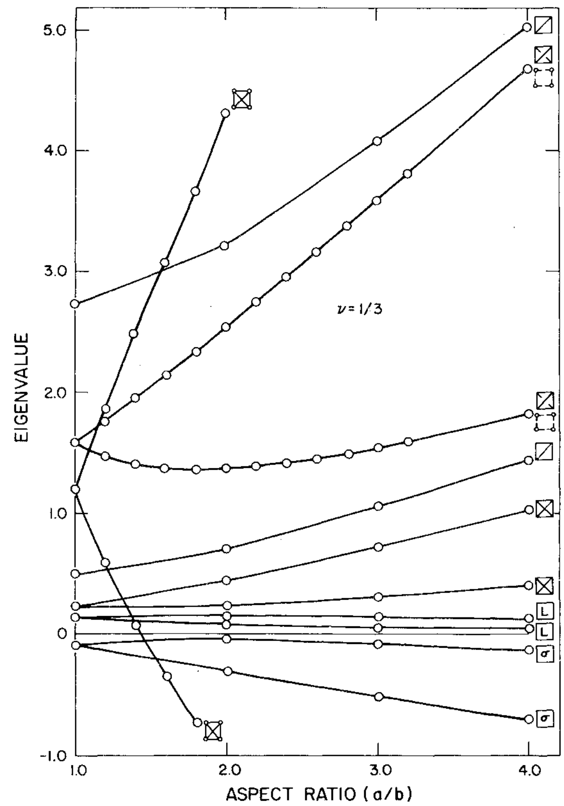

Fig. 2 Test results.

of 1/3. Also the results are contingent on the specific element subdivision displayed in Fig. 1, although it is expected that the same conclusions would apply regardless of element subdivision.

The words "upper bound" in Table 2, indicate that the element strain energy is always an upper bound on the strain energy of the corresponding exact solution. For element models which are absolutely convergent and independent of element size monotonicity is a necessary and sufficient condition for an upper bound on the strain energy. Elements developed using either a Ritz method or some form of a finite difference method are absolutely convergent if constant strain states exist. Of the elements tested only the Turner stress rectangle is not associated with either of these methods and the absolute convergence of this element has not been proven. As the test suggests and has been previously established, all the remaining element models possess constant strain states. Moreover, all of the models are

Table 2 Summary of test results

Symbol	Known characteristics	Present test results
⊠	degenerate at $a/b = (3)^{1/2}$, $1/(3)^{1/2}$	monotone, upper bound for $1 \leq a/b \leq 1.44$
⧄	monotone, upper bound	monotone, upper bound
⊠	monotone, upper bound	monotone, upper bound
[σ]	...	monotonicity not guaranteed
[L]	monotone, upper bound	monotone, upper bound
⊠	upper bound	monotone, upper bound
⁚⁚	...	monotone, upper bound

independent of element size. Thus, for six of the seven elements monotonicity is a necessary and sufficient condition for an upper bound on the strain energy.

As shown in Table 2, the known characteristics of the three Ritz elements were verified by the present test. Two of the non-Ritz elements, the unbiased triangle and the finite difference models were proven to be monotone and upper bound. As indicated in Table 2, the monotonicity of the unbiased triangle and both the monotonicity and upper boundedness of the finite difference model are new results. It is interesting to note that the unbiased triangle and the finite difference models turned out to be identical elements. A third non-Ritz model, the Hrennikoff frame was shown to be monotone and upper bound, dependent on panel aspect ratio. This is explained by the element degeneracy for panel aspect ratios greater than $(3)^{1/2}$ and less than $1/(3)^{1/2}$. The fact that the test predicts monotonicity for aspect ratios up to 1.44 rather than $(3)^{1/2}$ arises because of the constraints applied at the midside nodes to make the tests problem independent. The results for the Hrennikoff frame are also new. The final non-Ritz model, the Turner stress rectangle, failed the test for all aspect ratios tested. Since the shape functions of this element depend explicitly on Poisson's ratio the test was also applied for Poisson's ratios ranging from −0.9–0.4. In all cases the element failed the test. These results do not contradict any previously established results.

Convergence rates are indicated by the magnitudes of the nonzero eigenvalues of an element, the greater the magnitude the higher the expected rate of convergence (note that higher convergence rates do not imply better answers). For example, referring to Fig. 2, the expected convergence rate of the finite difference model is higher than that of Lagrange rectangle but not necessarily higher than that of the two triangle model. As shown in Fig. 2, convergence rates can vary strongly with panel aspect ratio. Moreover for most of the models, the convergence rate appears to be monotone with aspect ratio.

Conclusions

This paper presents a sufficiency test to guarantee monotonic convergence of particular finite element models. The test has the following salient features. 1) It is a numerical test. (Thus it is automatible, proves convergence for specific values of variables tested, and is suitable for any element shape and material constants.) 2) It is applicable to all types of displacement models. (The test is suitable for Ritz and non-Ritz elements, elements developed using shape functions or finite differences, and elements for vector or scalar field problems.) 3) It yields secondary information on convergence characteristics. (It provides measures of the convergence rate of the element and a necessary check on the existence of constant strain states and thereby on absolute convergence.)

Use of the test for seven rectangular membrane models suggest the test is unrestricted enough to be generally useful on many existing models. In particular: 1) It was successful in exposing monotonic convergence for five of the seven elements and the range of geometrics for monotonic convergence of a sixth. 2) It proved that monotonic convergence must be associated with three elements whose monotonicity was previously unproven. 3) The tests show that convergence rates of the elements may vary sharply as a function of panel aspect ratio. For a given element, extreme convergence rates are generally associated with square and extreme panel aspect ratios.

Though it can not be stated that elements that fail the test will not be associated with monotonic convergence, the general usefulness, ease of application, and large number of available elements suggest that elements that do not pass the test be avoided.

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